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Generalized $\mathscr{G}$-Theory

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A generalization of the $\mathscr{G}$-theory defined by A. Heil *et al.* is presented.

Various attempts to formulate the fundamental physical interactions in the framework of unified geometric theories have recently gained considerable success (Kaluza, 1921; Klein, 1926; Trautmann, 1970; Cho, 1975). Symmetries of the spacetime and so-called internal spaces seem to play a key role in investigating both the fundamental interactions and the abundance of elementary particles.

We would like to present a category-theoretic description of a generalization of the $\mathscr{G}$-theory concept and its application to geometric compactification and dimensional reduction (Heil *et al.*, 1987*a,b*). The main reasons for using categories and functors as tools are the clearness and the level of generalization we can obtain.

Introduction to $\mathscr{G}$-theories and category theory can be found in Heil *et al.* (1987*a,b*) and Bucur and Deleanu (1968), respectively.

Let us define a (generalized) $\mathscr{G}$-theory as a category C^u_Δ of quadruples $(U, F, \Delta\Gamma, G)$, where G is a symmetry group, $F \xrightarrow{\pi} U$ is a G-bundle over the G-space (manifold) U (Heil *et al.*, 1987*a,b*; Herrlich and Strecker, 1973) $[\pi(vg) = \pi(v)g]$, and $\Delta\Gamma$ is the set of sections of the bundle $F \xrightarrow{\pi} U$ with a G-property Δ (Heil *et al.*, 1987*a,b*).

A morphism between two objects of C^u_Δ is a triple $(f, \tilde{f}, h)$, where

$$
\begin{aligned}
h&: \quad G \to G' \text{ group homomorphism} \\
f&: \quad U \to U \\
\tilde{f}&: \quad F \to F'
\end{aligned}
$$

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and the following conditions are fulfilled:

(i) $f(ug) = f(u)h(g)$.
(ii) $\tilde{f}(vg) = f(v)h(g)$.
(iii) $\pi' \circ \tilde{f} = f \circ \pi$.

The simplest case when C^u_Δ consists of only one object $(U, F, \Delta\Gamma, G)$ (morphism = identity map) corresponds to the $\mathcal{G}$-theories considered in Heil *et al.* (1987*a,b*). There is of course a natural generalization with symmetries of $(U, F, \Delta\Gamma, G)$ as morphism. The generalization of $\mathcal{G}$-theory described above gives one leeway to investigate unification and hierarchies without leaving the basic category. Of course, we should answer the question: when are two $\mathcal{G}$-theories physically equivalent? The answer follows:

Definition 1. We say that two $\mathcal{G}$-theories are isomorphic if they are isomorphic as categories (Bucur and Deleanu, 1968). ■

This is not the notion of equivalence we are looking for. Let us denote by $C(X, Y)$ the set of the morphism $X \to Y$ of a category C.

Definition 2. A subcategory DC of a category C is called a skeleton if it contains one and only one representative of each equivalence class of objects of C and $DC(X, Y) = C(X, Y)$. ■

Definition 3. Two categories C and C' are said to be equivalent if there are two functors $F: C \to C'$ and $G: C' \to C$ such that $G \cdot F$ and $F \cdot G$ are isomorphic to the identity functors id_C and $\mathrm{id}_{C'}$, respectively. ■

The functor F (G) is called an equivalence of categories C and C' $(C'$ and $C)$.

Proposition 1. The inclusion $I: DC \to C$ is an equivalence. ■

Proposition 2. Every category has a skeleton. Any two skeletons of a given category C are isomorphic. ■

Proposition 3. Two categories are equivalent if and only if their skeletons are isomorphic. ■

Proposition 4. A functor $F: C \to C'$ is an equivalence if and only if it is bijective and each $X \in \mathrm{Ob}\, C'$ is isomorphic to $F(Y) \in \mathrm{Ob}\, C'$ for some $Y \in \mathrm{Ob}\, C$. ■

Now we are ready to state:

Definition 4. Two $\mathcal{G}$-theories are said to be equivalent if and only if they are equivalent as categories. ■

In this way we have:

Definition 5. A category $\tilde{C}^u_\Delta$ is said to be a generalized $\mathscr{G}$-theory if and only if it is equivalent to a $\mathscr{G}$-theory C^u_Δ. ■

The constructions described in Heil *et al.* (1987*a,b*) take the following form in our approach. A dimensional reduction of a $\mathscr{G}$-theory C^u_Δ to a $\mathscr{G}$-theory C^U is a functor $F: C^u_\Delta \to C^{U_{\text{eff}}}_{\Delta_{\text{eff}}}$, where Δ_{eff} is a low-dimensional (physical) manifestation of the property Δ, and U_{eff} is an effective spacetime. Our intuition requires that U_{eff} should be in some way related to U, but the formalism does not demand this. We do not require F to be an equivalence, but in some constructions it may be desirable.

Theorem 1. Let F be a functor $C_0 \to T$. There exists a category C, an equivalence $G: C_0 \to C$, and such a functor $F': C \to T$ so that:

(i) $F = F'G$.
(ii) For any two $Y \in \text{Ob}\, C$, $X \in \text{Ob}\, T$ and any T-isomorphism $f: X \to F(Y)$ there exists a C-isomorphism $g: Y \to Y'$ so that $F'(g) = f$.

Proof. Let us construct the category C in the following way. The objects are the pairs (f, X), where f is a T-isomorphism $Y' \to X'$ and $X \in \text{Ob}\, C_0$, so that $F(X) = X'$. A morphism $(f, X) \to (\tilde{f}, \tilde{X})$ is a triple $(h, f, \tilde{f})$, where h is any C_0-morphism. The composition law is defined by

$$(h', g, \tilde{g}) \cdot (h, f, \tilde{f}) = (h'h, f, \tilde{g})$$

The functor G transforms X into (id_x, X) and a morphism $f: X \to Y$ into $(f, \text{id}_x, \text{id}_y)$. The functor F' transforms (f, X), where f: $Y' \to X$, into Y' and a morphism $(h, f, \tilde{f})$ into $\tilde{f}^{-1} \cdot F(h) \cdot f$. ■

Proposition 5. Symmetries (isomorphisms) are "lifted" by dimensional reduction functors.

Proof. See Theorem 1. ■

A restriction of $\mathscr{G}$-theory described in Heil *et al.* (1987*a,b*) has an obvious generalization in our approach. As in Heil *et al.* (1987*a,b*), the restricted $\mathscr{G}$-theory need not be equivalent to the initial one. In fact, it may have a richer physical structure. There is much work to be done to understand such physical constructions as compactification and dimensional reduction. We hope that the general approach suggested above will make it easier to investigate universal aspects of these constructions. Some of them are under investigation and will be discussed in a subsequent paper.

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